# ON THE STABILITY OF VIBRATIONALLY LINEARIZED NONLINEAR SYSTEMS

## (OB USTOICHIVOSTI VIBRATSIONNO-LINEARIZIROVANNYKH NELINEINYKH SISTEM)

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Many problems of mechanics and automatic control lead of necessity to the consideration of motions of systems containing nonlinear elements, with discontinuous characteristics and being subject to the action of periodically changing perturbations, the frequencies of which are considerably higher than those of the basic steady or quasi-stationary motion of the system.

In such cases the equations of motion admit, as a rule, a solution which consists of two terms: the first term corresponds to the basic (slow) motion, while the second contains high-frequency components, the amplitudes of which are relatively small in comparison with the term corresponding to the basic motion.

In order to obtain the differential equations describing the variation of the slow component, procedures have been used which are known as the methods of vibrational linearization or vibrational smoothing [1-3 ]. Recently, by means of these methods, results have been obtained [2-6 ] which are of primary importance for applications.

Popov [6 ] proposed the use of vibrationally linearized equations for determining not only the basic slow motion of the system but also its stability.

Without trying to give a rigorous foundation to the method under consideration, in the present paper the authors indicate the existence of a certain condition to be satisfied in order that the arguments on which this method is based be correct. The fact is that the basic motion of the system can be a slow one, while the small deviations from that motion caused by the perturbations can turn out to be rapid motions for the description of which the vibrationally linearized equations are useless. Therefore, using the vibrationally linearized equations in a stability problem, it is necessary to verify that the non-stationary motions described by these equations are also sufficiently slow in comparison with the speed of variation of the vibrational component conditioning the linearization.

**1.** Consider a system, the differential equation of motion of which is of the form

$$Q(p)x + R(p)F(x) = S_1(p)f_1(t) + S_2(p)f_2(t) \tag{1.1}$$

where $Q(p)$, $R(p)$, $S_1(p)$ and $S_2(p)$ are differential operators which in turn are polynomials of the operator $p = d/dt$; $t$ is the time, $x$ a generalized coordinate and $F(x)$ a nonlinear function.

Let the external action be a function of $t$, say $f_1(t)$, which is characterized by the frequency $\omega$, the latter being considerably smaller than the frequency $\Omega$ of the function $f_2(t)$ which, for reasons of simplicity, is assumed to be harmonic, i.e.

$$f_2(t) = B \sin \Omega t \tag{1.2}$$

In order to solve the problem approximately we usually proceed as follows [2]. We seek this solution in the form

$$x(t) = x^\circ(t) + x^*(t) \tag{1.3}$$

where $x^*(t) = A \sin(\Omega t + \phi)$; $x^\circ(t)$ is a slowly varying component ($A$ and $\phi$ are slowly changing functions of time).

Substituting (1.3) into (1.1) we obtain

$$Q(p)x^\circ + R(p)F(x^\circ + x^*) + Q(p)x^* = S_1(p)f_1(t) + S_2(p)f_2(t) \tag{1.4}$$

Expanding $F(x^\circ + x^*)$ into its Fourier series according to the harmonics $\sin k(\Omega t + \phi)$ and $\cos k(\Omega t + \phi)$ we have, provided that we limit ourselves to the terms with zero- and first-order harmonics (it is assumed that higher harmonics of the system are not left out and, consequently, their omission does not cause any gross errors):

$$F(x) = F^\circ(x^\circ, A) + q(x^\circ, A)x^* + \frac{q'(x^\circ, A)}{\Omega} px^* + \cdots \tag{1.5}$$

where

$$F^\circ(x^\circ, A) = \frac{1}{2\pi} \int_0^{2\pi} F(x^\circ + A \sin\psi)\, d\psi$$

$$q(x^\circ, A) = \frac{1}{\pi A} \int_0^{2\pi} F(x^\circ + A \sin\psi) \sin\psi \, d\psi$$

$$q'(x_0, A) = \frac{1}{\pi A}\int_0^{2\pi} F(x^\circ + A\sin\psi)\cos\psi d\psi \tag{1.6}$$

Taking (1.5) and (1.6) into account we obtain from (1.4) two differential equations. One of them determines the basic slow motion while the other determines the rapidly changing component of the motion:

$$Q(p)x^\circ + R(p)F^\circ(x^\circ, A) = S_1(p)f_1(t) \tag{1.7}$$

$$Q(p)x^* + R(p)\left(qx^* + \frac{q'}{\Omega}px^*\right) = S_2(p)f_2(t) \tag{1.8}$$

Equation (1.8) determines the amplitude $A$ and the phase $\phi$ as functions of $x^\circ$. Then, by means of the obtained relation $A = A(x^\circ)$, the amplitude $A$ is eliminated from the expression of the function $F^\circ(x^\circ, A)$. Usually the function $\Phi(x^\circ) = F^\circ[x^\circ, A(x^\circ)]$ is linearized, and for all odd symmetrical nonlinearities is finally obtained in the form

$$F^\circ = \Phi(x^\circ) \approx k^\circ x^\circ \tag{1.9}$$

As a result of this the linearized equation (1.7) for the slowly changing component assumes the form

$$[Q(p) + k^\circ R(p)]x^\circ = S_1(p)f_1(t) \tag{1.10}$$

where the coefficient $k^\circ$ depends, of course, on the amplitude $B$ and the frequency $\Omega$ of the high-frequency action.

2. Consider, now, the problem of investigating the stability of the solution $x^\circ(t) + x^*(t)$. Assume for this purpose the existence of a perturbation $\xi(t)$. If this perturbation is a slowly changing function, then it is natural to associate this function with the term $x^\circ(t)$. In such a case we can actually conclude that the equation for $x^\circ(t)$ permits us to decide the stability of the motion. This is done, for example, in [4-6].

Generally speaking, however, $\xi(t)$ can turn out to be a rapidly changing function. Then, naturally, on the basis of study of the linearized equation, an authentic judgement concerning the stability cannot be obtained, the reason being that this equation is valid only for motions which are sufficiently slow in comparison with the rate of change of the high-frequency component. It is possible that the non-stationary motions described by the vibrationally linearized equations turn out to be rapid motions even if the usual requirement concerning the filter properties of the high frequencies of the linear part of the system is satisfied, i.e. under the condition of sufficient slowness of the free vibrations of the linear part of the system. The above will be illustrated by the example given below.

**3.** Consider a system with one degree of freedom, the motion of which is described by the equation

$$\ddot{x} + k^2 x + F(x) = B_0 \sin \omega t + B \sin \Omega t \tag{3.1}$$

where

$$F(x) = \begin{cases} C & x > 0 \\ 0 & x = 0 \\ -C & x < 0 \end{cases} \tag{3.2}$$

Let $\Omega \gg \omega$. Applying the above method we obtain

$$F^\circ \approx \frac{2C}{\pi A} x^\circ, \qquad q = \frac{4C}{\pi A}, \qquad q' = 0 \qquad (|x^\circ| < |A|) \tag{3.3}$$

Equations (1.7) and (1.8) for the slow and rapid components of motion are

$$\ddot{x}^\circ + \left(k^2 + \frac{2C}{\pi A}\right) x^\circ = B_0 \sin \omega t \tag{3.4}$$

$$\ddot{x}^* + \left(k^2 + \frac{4C}{\pi A}\right) x^* = B \sin \Omega t \tag{3.5}$$

respectively. Seeking the solution of Equation (3.5) in the form $x^* = A \sin(\Omega t + \phi)$ we obtain

$$A = \frac{B - 4C/\pi}{k^2 - \Omega^2}, \qquad \varphi = 0 \tag{3.6}$$

Then Equation (3.4) assumes the form

$$\ddot{x}^\circ + \left[k^2 - \frac{k^2 - \Omega^2}{2(1 - \pi B/4C)}\right] x^\circ = B_0 \sin \omega t \tag{3.7}$$

From here the slow motion we are seeking is obtained in the form

$$x^\circ = \frac{B_0}{k^2 - \omega^2 - \dfrac{k^2 - \Omega^2}{2(1 - \pi B/4C)}} \sin \omega t \tag{3.8}$$

The last formula, by virtue of (3.3) and (3.6), holds only if the inequality

$$\left| \frac{B_0}{k^2 - \omega^2 - \dfrac{k^2 - \Omega^2}{2(1 - \pi B/4C)}} \right| < \left| \frac{B - 4C/\pi}{k^2 - \Omega^2} \right| \tag{3.9}$$

is satisfied. The latter, however, holds always provided that $B_0$ is sufficiently small.

It is not difficult to see that in spite of the slowness of the motion (3.8), and also of the slowness of the free vibrations of the system without the nonlinear element, the free vibrations described by Equation

(3.7) can turn out to be relatively "rapid", the reason for this being that their frequency

$$\lambda^2 = k^2 - \frac{k^2 - \Omega^2}{2(1 - \pi B / 4C)} \tag{3.10}$$

is sometimes comparable (or even higher) with the frequency $\Omega$ of the rapid motion. In fact, let $k = 1$ sec $^{-1}$, $\omega = 10$ sec$^{-1}$, $\Omega = 100$ sec$^{-1}$, $B = 1$, $C = 1/2\pi$. Then

$$\lambda \approx \sqrt{1 + 10^4} \approx 10^2 \sec^{-1} = \Omega$$

In such a case, of course, we cannot expect that Equation (3.4) will correctly describe even the small deviations from the basic slow motion. In fact, neglecting in (3.10) the square of the frequency $k$ in comparison with the square of the frequency $\Omega$, and assuming that the second term in Formula (3.10) is much larger than $k^2$, we obtain the relation

$$\lambda = \frac{\Omega}{\sqrt{2(1 - \pi B / 4C)}} \tag{3.11}$$

from which, by virtue of the assumptions made, follows the stability condition in the form of the inequality

$$1 - \frac{\pi}{4}\frac{B}{C} > 0 \tag{3.12}$$

Formula (3.11) shows that near the "boundary of the stability" $1 - \pi B/4C = 0$, determined by the vibrationally linearized equation, the frequency of the perturbed motion $\lambda$ is very large. Therefore, for the example under consideration, the condition of stability (3.12) cannot be considered as even vaguely proved.

The result obtained convincingly shows the necessity of the above-formulated requirement for the sufficient slowness of the perturbed motion in comparison with the rapidity of the change of the high-frequency component.

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